Effect of viscosity on wave propagation in anisotropic thermoelastic medium with three-phase-lag model

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Abstract

The aim of the present paper is to study the wave propagation in anisotropic viscoelastic medium in the context of the theory three-phase-lag model of thermoelasticity. It is found that there exist two quasi-longitudinal waves (qP1, qP2) and two transverse waves (qS1, qS2). The governing equations for homogeneous transversely isotropic thermoviscoelastic are reduced as a special case from the considered model. Different characteristics of waves like phase velocity, attenuation coefficient, specific loss and penetration depth are computed from the obtained results. Viscous effect is shown graphically on different resulting quantities for two-phase-lag model and three-phase-lag model of thermoelasticity. Some particular cases of interest are also deduced from the present investigation.

Keywords: Wave propagation, Viscoelastic, Three-Phase-lag, Two-Phase-lag model, Anisotropic.

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1   Introduction

The generalized theory of thermoelasticity is one of the modified version of classical uncoupled and coupled theory of thermoelasticity and have been developed in order to remove the paradox of physical impossible phenomena of infinite velocity of thermal signals in the classical coupled thermoelasticity. Hetnarski and Ignaczak [1] examined five generalizations of the coupled theory of thermoelasticity.

The first generalization is due to Lord and Shulman [2] who formulated the generalized thermoelasticity theory involving one thermal relaxation time. This theory is referred to as L-S theory or extended thermoelasticity theory in the Maxwell-Cattaneo law replaces the Fourier law of heat conduction by introducing a single parameter that acts as a relaxation time,who obtained a wave-type equation by postulating a new law of heat conduction instead of classical Fourier's law. Green and Lindsay [3] developed a temperature rate- dependent thermoelasticity that includes two thermal relaxation times and does not violate the classical Fourier's law of heat conduction, when the body under consideration has a center of symmetry. One can refer to Hetnarski and Ignaczak [4] for a review and presentation of generalized theories of thermoelasticity. Chadwick [5-6] discussed propagation of plane harmonic waves in transversely isotropic and homogeneous anisotropic heat conduction solids respectively. Sharma *et al.* [7-9] studied the wave propagation in anisotropic solids in generalized theory of thermoelasticity. Sharma [10] discussed the existence of longitudinal and transverse in anisotropic thermoelastic media. The third generalization of the coupled theory of thermoelasticity is developed by Hetnarski and Ignaczak and is known as low-temperature thermoelasticity. The fourth generalization to the coupled theory of thermoelasticity introduced by Green and Naghdi and this theory is concerned with the thermoelasticity theory without energy dissipation, referred to as G-N theory of type II in which the classical Fourier law is replaced by a heat flux rate-temperature gradient relation. The heat transport equation does not involve a temperature rate term and as such this model admits undamped thermoelastic waves in thermoelastic material. The fifth generalization of the coupled theory of thermoelasticity is developed by Tzau [11] and Chandrasekhariah [12] and is referred to dual phase- lag thermoelasticity. Raychoudhuri [13] has recently introduced the three-phase-lag heat conduction equation in which the Fourier law of heat conduction is replaced by an approximation to a modification of the Fourier law with the introduction of three different phase-lags for the heat flux vector, the temperature gradient

and the thermal displacement gradient. The stability of the three-phase-lag heat conduction equation is discussed by Quintanilla and Racke [14].Quintanilla has studied the spatial behavior of solutions of the three-phase-lag heat conduction equation. Subsequently Kar and Kanoria [15] have employed this theory of thermoelasticity with three-phase-lag to discuss a problem of thermoelastic interactions on functional graded orthotropic hollow sphere under thermal shock. Mukhopadhyay and Kumar [16] has analyzed the effects of phase-lags on wave propagation in a thick plate under axisymmetric temperature distribution. The linear thermoviscoelastic remains an important area of research not only due to the advent and use ogf polymer, but also because most solids when subjected to dynamic loading exhibit viscous effects. Kumar [17] discussed wave propagation in mocropolar viscoelastic generalized thermoelastic solid. El-Karamany [18] studied uniqueness and reciprocity theorems in a generalized in linear micropolar themoelasticity. Simonetti [19] investigated Lamb wave propagation in elastic plates coated with viscoelastic materials. Sharma [20] discussed the problem of Rayleigh-Lamb wave propagation in visco-thermoelastic plate. Baksi et al.[21] discussed the two-dimensional visco-elastic problems in generalized thermoelastic medium with heat source. Sharma *et al.* [22] investigated the Lamb wave's propagation in viscothermoelastic plate under fluid loadings. Kumar and Partap [23] discussed the vibration analysis of wave micropolar thermoviscoelasic plate. Kumar and chawla [24] discussed the plane wane propagation in anisotropic three phase lag model and two-phase lag model.

Keeping in view of these applications, we studied the propagation of waves in the context of three-phase-lag of medium, for anisotropic thermoviscoelastic medium. As a special case, the basic equations for homogeneous transversely isotropic thermoelastic three-phase lag are reduced. Viscous effect is shown graphically on different characteristics of waves like phase-velocity; attenuation coefficient, specific loss and penetration depth.

2 Fundamental equations

The basic equations for homogeneous anisotropic thermoelastic solid, without body forces and heat sources are given as

Constitutive relations

$$\sigma_{ij} = c_{ijkl}e_{kl} - \beta_{ij}T, \quad \beta_{ij} = c_{ijkl}\alpha_{kl}, \tag{1}$$

$$\rho S T_0 = \rho C^* T + \beta_{ij} T_0 e_{ij}, e_{ij} = (u_{i,j} + u_{j,i})/2. \tag{2}$$

Equations of motion in the absence of body force

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{3}$$

The energy equation (without extrinsic heat supply) is

$$\rho \dot{S} T_0 = -q_{i,i}. \tag{4}$$

The Fourier law (for thermoelastic three- phase- lag model) is given by Roy Choudhuri [11] as

$$q_i = -[K_{ij} T_{,j}(P, t + \tau_t) + K_{ij}^* v_{,j}(P, t + \tau_v)]. \tag{5}$$

Here $c_{ijkm}(= c_{kmij} = c_{ijkm} = c_{ijmk})$ are elastic parameter; $\dot{v} = T$, $\vec{u}$ is the displacement vector, c_{ijkl} are the elastic parameter, β_{ij} are the tensor of thermal respectively. ρ and C^* are density and specific heat at constant strain; T_0 is the reference temperature assumed to be such that $\left|\frac{T}{T_0}\right| << 1$. q_i, S is the heat flux vector and entropy per unit mass respectively. $T(x_1, x_2, x_3, t)$ is the temperature distribution from the reference temperature T_0; $\sigma_{ij}(= \sigma_{ji})$, $K_{ij}(= K_{ji})$, $K_{ij}^*(= K_{ji}^*)$, e_{ij} are the components of stress, thermal conductivity, material constants characteristic of the theory and strain tensor respectively.

In the above equations symbol ("",") followed by a suffix denotes differentiation with respect to spatial coordinate and a superposed dot ("".") denotes the derivative with respect to time respectively.

3   Formulation of the problem

We consider a homogeneous, thermally conducting, anisotropic viscoelastic solid in the undeformed state at the uniform temperature T_0.

In order to account for the material damping behavior the material coefficient c_{ijkl} are assumed to be function of the time operator $D = \frac{\partial}{\partial t}$, i.e.

$$c_{ijkl} = \bar{c}_{ijkl},$$

$$\bar{c}_{ijkl} = c_{ijkl}(D). \tag{6}$$

Assumed that the viscoelastic nature of the material is described by the Voigt model of linear viscoelasticity (Kaliski[25]), we write

$$\bar{c}_{ijkl} = c_{ijkl}(1 + \tau \frac{\partial}{\partial t}). \tag{7}$$

The general system of equations for anistropic thermoviscoelastic material are obtained by using equation (1), (2) and (5),in equation (3) and (4), and with the aid of equation (7), the equation of motion and heat conduction are

Equations of motion

$$\bar{c}_{ijkl}e_{kl,j} - \beta_{ij}T_{,j} = \rho\ddot{u}_i. \tag{8}$$

Equation of heat conduction

$$K_{ij}\left(1 + \tau_T\frac{\partial}{\partial t}\right)\dot{T}_{,ji} + K_{ij}^*\left(1 + \tau_v\frac{\partial}{\partial t}\right)T_{,ji} =$$

$$\left(1 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2!}\frac{\partial^2}{\partial t}\right)\left[v_1^2(\rho C^*\ddot{T} + \beta_{ij}T_0\ddot{e}_{ij})\right], \tag{9}$$

We define the dimensionless quantities:

$$x_i' = \frac{\omega_1^*x_i}{v_1}, \quad u_i' = \frac{\omega_1^*u_i}{v_1}, \quad T = \frac{T}{T_0}, \quad \tau_T' = \omega_1^*\tau_T,$$

$$\tau_v' = \omega_1^*\tau_v, \quad \tau_q' = \omega_1^*\tau_q, \quad v_1^2 = \frac{\bar{c}_{1111}}{\rho}, \quad \omega_1^* = \frac{\rho C^*v_1^2}{K_{11}}. \tag{10}$$

Here ω_1^* is the characteristic frequency of the medium, v_1 is the longitudinal wave velocity in the isotropic version of the medium.

4   Solution of the problem

Using the dimensionless quantities defined by equation (10) in equations (8) - (9), after suppressing the primes and assuming the solution of the resulting equations as

$$(u_1, u_2, u_3, T) = (U_1, U_2, U_3, T^*)\exp[i(\xi x_m n_m - \omega t)], \tag{11}$$

where ω is the circular frequency and ξ is the complex wave number. U_1, U_2, U_3 and T^* are undetermined amplitude vectors that are independent of time t and coordinates x_i. n_m is the unit wave normal vector, we obtain

$$[\bar{c}_{ijkl}n_l n_j \xi^2 - \rho v_1^2 \omega^2 \delta_{ik}]U_k + i\xi T_0 \beta_{ij} n_j T^* = 0, \tag{12}$$

$$i\xi\beta_{ij}n_j v_1^2\omega^2\tau_{qq}^1 U_k + [\xi^2(i\omega\omega_1^* K_{ij}n_i n_j\tau_T^{11} - \tau_v^{11}K_{ij}^*n_i n_j)$$

$$+ \omega^2\rho v_1^2\tau_q^{11}]T^* = 0, \quad (13)$$

with

$$\tau_T^{11} = 1 - i\omega\tau_T, \quad \tau_v^{11} = 1 - i\omega\tau_v, \quad \tau_q^{11} = 1 - i\omega\tau_q - \frac{\tau_q^2}{2}\omega^2.$$

Equations (12)-(13) constitute the linear system of four homogeneous equations in four unknowns U_1, U_2, U_3 and T^*.

The Christoffel's tensor notation may be expressed as follows

$$\bar{\gamma}_{ij} = \bar{c}_{ijkl}n_l n_k, \quad \beta_i = \beta_{ij}n_j, \quad K = K_{ij}n_i n_j, \quad K^* = K_{ij}^* n_i n_j. \tag{14}$$

Using (14) in equations (12)-(13), we obtain

$$[\bar{\gamma}_{ij}\xi^2 - \rho\omega^2\delta_{ik}v_1^2]U_k + i\xi T_0\beta_i T^* = 0, \tag{15}$$

$$i\xi\omega^2 v_1^2\tau_q^{11}U_k + [\xi^2(i\omega\omega_1^* K\tau_T^{11} - \tau_v^{11}K^*) + \omega^2\rho C^* v_1^2\tau_q^{11}]T^* = 0. \tag{16}$$

The non-trivial solution of the system of equations (15)-(16) is ensured by the determinant equation

$$\begin{vmatrix} \bar{\gamma}_{11}\xi^2 - \rho\omega^2 v_1^2 & \bar{\gamma}_{12}\xi^2 & \bar{\gamma}_{13}\xi^2 & i\xi T_0\bar{\beta}_1 \\ \bar{\gamma}_{21}\xi^2 & \bar{\gamma}_{22}\xi^2 - \rho\omega^2 v_1^2 & \bar{\gamma}_{23}\xi^2 & i\xi T_0\bar{\beta}_2 \\ \bar{\gamma}_{31}\xi^2 & \bar{\gamma}_{32}\xi^2 & \bar{\gamma}_{33}\xi^2 - \rho\omega^2 v_1^2 & i\xi T_0\bar{\beta}_3 \\ i\xi\omega^2\tau_q^{11} & ir_1\xi\omega^2\tau_q^{11} & ir_2\xi\omega^2\tau_q^{11} & -r_3\xi^2 + r_4\omega^2 \end{vmatrix} \tag{17}$$

The equation (17) yields to the following polynomial equation in ξ as

$$A\xi^8 + B\xi^6 + C\xi^4 + D\xi^2 + E = 0. \tag{18}$$

The coefficients A, B, C, D, E are given as:

$$A = r_3(\bar{\gamma}_{11}f_1 + \bar{\gamma}_{12}f_2 + \bar{\gamma}_{13}f_3),$$

$$B = \rho v_1^2 r_3\omega^2 f_4 - r_4(\bar{\gamma}_{11}f_1 + \bar{\gamma}_{12}f_2 + \bar{\gamma}_{13}f_3)\omega^2 + T_0\omega^2\tau_q^{11}(\beta_1 f_5 + \bar{\gamma}_{13}f_6 + f_7),$$

$$C = \rho v_1^2\omega^4\left\{r_4 f_9 + f_{10}T_0 + \tau_q^{11}T_0(\beta_1 f_{11} + f_{12}) + \bar{\gamma}_{13}f_{13} - \rho v_1^2 f_{14}\right\},$$

$$D = \rho^2 v_1^4\omega^4\left\{f_{15} + \omega^2(\tau_q^{11}T_0 f_{16} + f_{17})\right\}, \qquad E = \omega^8 f_{18},$$

$$f_1 = \bar{\gamma}_{23}\bar{\gamma}_{32} - \bar{\gamma}_{22}\bar{\gamma}_{33}, \qquad f_{10} = \bar{\gamma}_{11}(r_2\beta_3 - r_1\beta_2) + \bar{\gamma}_{12}(r_4\bar{\gamma}_{21} + \beta_2\tau_q^{11}),$$

$$f_2 = \bar{\gamma}_{21}\bar{\gamma}_{33} - \bar{\gamma}_{23}\bar{\gamma}_{31}, \qquad f_{11} = (r_1\bar{\gamma}_{21} + r_2\bar{\gamma}_{31}) + (\bar{\gamma}_{21} + \bar{\gamma}_{33}),$$

$$f_3 = \bar{\gamma}_{31}\bar{\gamma}_{22} - \bar{\gamma}_{32}\bar{\gamma}_{21}, \qquad f_{12} = (r_2\bar{\gamma}_{32} - r_1\bar{\gamma}_{33})\beta_2 + (r_1\bar{\gamma}_{23} - r_2\bar{\gamma}_{22})\beta_3,$$

$$f_4 = (\bar{\gamma}_{11}\bar{\gamma}_{22} + \bar{\gamma}_{11}\bar{\gamma}_{33}) - (\bar{\gamma}_{12}\bar{\gamma}_{21} + \bar{\gamma}_{13}\bar{\gamma}_{31}) - f_1, \quad f_{13} = r_4\bar{\gamma}_{31} + \beta_3,$$

$$f_5 = (r_2\bar{\gamma}_{32} - r_1\bar{\gamma}_{31}) + \bar{\gamma}_{23}(r_1\bar{\gamma}_{31} - \bar{\gamma}_{32}) + \bar{\gamma}_{21}(\bar{\gamma}_{33} - r_2\bar{\gamma}_{31}), \quad f_{14} = r_3(\bar{\gamma}_{11} + \bar{\gamma}_{33}),$$

$$f_6 = r_1(\beta_3\bar{\gamma}_{21} - \beta_2\bar{\gamma}_{31}) + (\beta_2\bar{\gamma}_{32} - \beta_3\bar{\gamma}_{22}), \quad f_{15} = r_4\omega^2(\bar{\gamma}_{11} + \bar{\gamma}_{22} + \bar{\gamma}_{33}) - \bar{\gamma}_{22}r_3,$$

$$f_7 = r_{11}\bar{\gamma}_{11} + r_{12}\bar{\gamma}_{12}, \qquad f_{16} = (r_1\beta_2 - r_2\beta_3) + \beta_1,$$

$$f_8 = (\bar{\gamma}_{11}\bar{\gamma}_{22} + \bar{\gamma}_{11}\bar{\gamma}_{33}), \qquad f_{17} = \rho v_1^2 r_3,$$

$$f_9 = f_1 - f_8, \qquad f_{18} = -\rho^3 v_1^6 r_4,$$

$$r_1 = \frac{\beta_2}{\beta_1}, \quad r_2 = \frac{\beta_3}{\beta_1}, \quad r_3 = \frac{\bar{K}^*\tau_v^{11} - i\omega\omega_1^*\bar{K}\tau_t^{11}}{\beta_1 v_1^2}, \quad r_4 = \frac{\rho C^*\tau_q^{11}}{\beta_1},$$

$$r_{11} = \bar{\gamma}_{11}[(r_1\bar{\gamma}_{33} - r_2\bar{\gamma}_{32})\beta_2 - (r_2\bar{\gamma}_{22} + r_1\bar{\gamma}_{23})\beta_3],$$

$$r_{12} = \bar{\gamma}_{12}[(\beta_3\bar{\gamma}_{23} - \beta_2\bar{\gamma}_{33})\beta_2 + (\beta_2\bar{\gamma}_{31} - \beta_3\bar{\gamma}_{21})r_2].$$

On solving equation (18), we obtain eight roots of ξ that is, $\pm\xi_1, \pm\xi_2, \pm\xi_3$ and $\pm\xi_4$ corresponding to these roots, there exists four waves corresponding to descending order of their velocities namely a quasi P-wave (qP1) and two quasi transverse(qS1,qS2) and a quasi-thermal wave (qP2).

The expressions of phase velocity, attenuation coefficient, specific loss and penetration depth of these types of waves are given in **Appendix A.**

5   Special cases

5.1   Three phase lag thermoelasticity

If we take

$$\tau \to 0, \tag{19}$$

then the above analysis is reduced to three phase lag model of thermoelasic. The above results are similar as those obtained by Kumar and Chawla [24].

5.2 Two-phase lag model

If we take $K_{ij}^* \to 0$, in the above analysis, we obtain result corresponding to the two-phase-lag model of viscothermoelastic solid.

5.3 Anisotropic viscoelastic media

In the absence of thermal effect, we obtain from equation (16), the polynomial equation corresponding to anisotropic viscoelastic medium as

$$\rho^3 V^6 - \rho(\bar{\gamma}_{11} + \bar{\gamma}_{22} + \bar{\gamma}_{33})V^4 + \rho\Gamma_1 V^2 + \Gamma_2 = 0. \tag{20}$$

Here $V = \frac{\omega}{\xi}$ is the wave velocity and

$$\Gamma_1 = \bar{\gamma}_{23}\bar{\gamma}_{32} + \bar{\gamma}_{12}\bar{\gamma}_{21} + \bar{\gamma}_{13}\bar{\gamma}_{31} - \bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{11}\bar{\gamma}_{33} - \bar{\gamma}_{22}\bar{\gamma}_{33},$$

$$\Gamma_2 = \bar{\gamma}_{13}(\bar{\gamma}_{21}\bar{\gamma}_{32} - \bar{\gamma}_{31}\bar{\gamma}_{22}) + \bar{\gamma}_{12}(\bar{\gamma}_{23}\bar{\gamma}_{31} - \bar{\gamma}_{21}\bar{\gamma}_{33}) + \bar{\gamma}_{11}(\bar{\gamma}_{22}\bar{\gamma}_{33} - \bar{\gamma}_{23}\bar{\gamma}_{32}).$$

As a special case, in the absence of viscosity effect, the equations (20) are the same as those obtained by Rose [26] for anisotropic elastic medium.

5.4 Transversely isotropic media:

Applying the transformation

$$x_1' = x_1 \cos\phi + x_2 \sin\phi, \quad x_2' = -x_1 \sin\phi + x_2 \cos\phi, \quad x_3' = x_3, \tag{21}$$

(ϕ is the angle of rotation in the $x_1 - x_2$ plane) in the equations (8)-(9), the basic equations for homogeneous transversely isotropic thermoviscoelastic three- phase- lag model are

$$\bar{c}_{11}u_{1,11} + \bar{c}_{12}u_{2,21} + \bar{c}_{13}u_{3,31} + \bar{c}_{66}(u_{1,22} + u_{2,12})$$
$$+ \bar{c}_{44}(u_{1,33} + u_{3,13}) - \beta_1 T_{,1} = \rho\ddot{u}_1, \tag{22}$$

$$\bar{c}_{66}(u_{1,21} + u_{2,11}) + \bar{c}_{12}u_{1,12} + \bar{c}_{11}u_{2,22}$$
$$+ \bar{c}_{44}u_{2,23} + (\bar{c}_{13} + \bar{c}_{44})u_{3,32} - \beta_1 T_{,2} = \rho\ddot{u}_2, \tag{23}$$

$$(\bar{c}_{13} + \bar{c}_{44})(u_{1,13} + u_{2,23}) + \bar{c}_{44}(u_{3,11} + u_{3,22})$$
$$+ \bar{c}_{33}u_{3,33} - \beta_3 T_{,3} = \rho\ddot{u}_3, \tag{24}$$

$$K_1\left(1+\tau_T\frac{\partial}{\partial T}\right)(\dot{T}_{,11}+\dot{T}_{,22})+K_3\left(1+\tau_T\frac{\partial}{\partial T}\right)\dot{T}_{,33}+$$

$$K_1^*\left(1+\tau_v\frac{\partial}{\partial T}\right)(T_{,11}+T_{,22})+K_3^*\left(1+\tau_v\frac{\partial}{\partial T}\right)T_{,33}=$$

$$\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)\rho C^*\ddot{T}+\left(1+\tau_q\frac{\partial}{\partial t}+\right.$$

$$\left.\tau_q^2\frac{\partial^2}{\partial t^2}\right)T_0\left[\beta_1(\ddot{u}_{1,1}+\ddot{u}_{2,2})+\beta_3\ddot{u}_{1,1}\right]. \tag{25}$$

Here $\quad \beta_{ij}=\beta_i\delta_{ij},\ \ K_{ij}=K_i\delta_{ij},\ \ K_{ij}^*=K_i^*\delta_{ij},\quad i$ is not summed,

$$\beta_1=(c_{11}+c_{12})\alpha_1+c_{13}\alpha_3,\ \ \beta_3=2c_{13}\alpha_1+c_{33}\alpha_3,$$

and α_1 and α_3 are the coefficients of linear thermal expansion.

In the above equations, we use the contracting notations $1\to11,\ 2\to22,\ 3\to33,\ 4\to23,\ 5\to31,\ 6\to12$ to relate c_{ijkm} to $c_{\varpi\vartheta}$ $\quad(i,j,k,m=1,2,3)$ and $(\varpi,\vartheta=1,2,3,4,5,6)$.

Applying the dimensionless quantities defined by (10) in equations (21)-(25) and using the solutions defined by (11), we obtain the following characteristic equations

$$A^*\xi^8+B^*\xi^6+C^*\xi^4+D^*\xi^2+E^*=0, \tag{26}$$

where

$$A^*=g_{14}(s_1g_1+s_4g_2+s_3g_3),$$

$$B^*=\Lambda_1g_1+\Lambda_2g_2+\Lambda_3g_3+\Lambda_4g_4+\Lambda_5\omega^2,$$

$$C^*=\omega^2(s_{16}g_1+s_{17}\omega^2+s_{18}+s_{19}+s_{20}+s_{21}),$$

$$D^*=\omega^4(s_{22}-s_{23}-g_{14}\omega^2),\qquad E^*=-\omega^6g_{13},$$

$$g_1=n_1^2+\delta_3n_2^2+\delta_4n_3^2,\quad g_6=\delta_2n_2n_3,$$

$$g_2=\delta_1n_1n_2\quad g_7=i\gamma_1n_2,$$

$$g_3=\delta_2n_1n_3,\quad g_8=(n_1^2+n_2^2)\delta_4+\delta_5n_3^2,$$

$$g_4=i\gamma_1n_1,\quad g_9=i\gamma_3n_3,$$

$$g_5=n_2^2+\delta_3n_1^2+\delta_4n_3^2,\qquad g_{10}=in_1\omega^2q_6^*,$$

$$g_{11}=in_2\omega^2q_6^*,\quad g_{12}=in_3\omega^2q_7^*,$$

$$g_{13}=q_5\omega^2,\quad g_{14}=i\omega[q_1^*(n_1^2+n_2^2)+q_2^*n_3^2]-q_3^*(n_1^2+n_2^2)-q_4^*n_3^2,$$

$$s_1 = g_5 g_8 - g_6 g_6, \quad s_2 = g_3 g_8 - g_2 g_8, \quad s_3 = g_2 g_6 - g_3 g_5,$$

$$s_4 = g_5 g_{14} + g_8 g_{14}, \quad s_5 = g_6 g_{11} - g_5 g_{12}, \quad s_6 = g_8 g_{11} - g_6 g_{12},$$

$$s_7 = g_9 g_{12} - g_8 g_{13}, \quad s_8 = g_3 g_{13} - g_9 g_{10}, \quad s_9 = g_8 g_{10} - g_3 g_{12},$$

$$s_{10} = g_6 g_{13} - g_9 g_{11}, \quad s_{11} = g_7 g_{11} - g_5 g_{13}, \quad s_{12} = g_4 g_{12} - g_6 g_{11},$$

$$s_{13} = g_6 g_6 - g_5 g_8 g_{10}, \quad s_{14} = g_2^2 g_{14}, \quad s_{15} = g_{10}(g_5 g_9 - g_6 g_7),$$

$$s_{16} = g_9 g_{12} - g_5 g_{13}, \quad s_{17} = g_{14}(g_1 + g_5 + g_8), \quad s_{18} = g_5(g_9 g_{12} - g_8 g_{13}),$$

$$s_{19} = g_{13}(g_6 g_6 - g_1 g_8), \quad s_{20} = g_{10}(g_4 g_8 - g_8 g_7), \quad s_{21} = g_3(g_3 g_{14} - g_4 g_{12}),$$

$$s_{22} = g_{13}(g_1 + g_5 + g_8) - (g_9 g_{12} - g_7 g_{11} + g_4 g_{10}),$$

$$\Lambda_1 = (s_1 g_{13} - s_4 \omega^2 + s_9 g_9 + s_6 g_7), \quad \Lambda_2 = s_7 g_2 + s_8 g_8 + s_9 g_7 + s_{10} g_3 + s_6 g_4,$$

$$\Lambda_3 = s_{11} g_3 + s_{15}, \quad \Lambda_4 = s_{12} g_3 + s_{13}, \quad \Lambda_5 = g_{14}(1 - s_1)\omega^2,$$

$$\delta_1 = \frac{\bar{c}_{12} + \bar{c}_{66}}{\bar{c}_{11}}, \quad \delta_2 = \frac{\bar{c}_{13} + \bar{c}_{44}}{\bar{c}_{11}}, \quad \delta_3 = \frac{\bar{c}_{66}}{\bar{c}_{11}},$$

$$\delta_4 = \frac{\bar{c}_{44}}{\bar{c}_{11}}, \quad \gamma_1 = \frac{\beta_1 T_0}{\bar{c}_{11}}, \quad \gamma_3 = \frac{\beta_3 T_0}{\bar{c}_{11}},$$

$$q_1^* = \frac{\bar{K}_1 \tau_t^1}{v_1^2}, \quad q_2^* = \frac{\bar{K}_3 \tau_t^1}{v_1^2}, \quad q_3^* = \frac{\bar{K}_1^* \tau_v^1}{\omega_1^* v_1^2}, \quad q_4^* = \frac{\bar{K}_3^* \tau_v^1}{\omega_1^* v_1^2},$$

$$q_5^* = \frac{\rho C^* \tau_q^1}{\omega_1^*}, \quad q_6^* = \frac{\beta_1 \tau_q^1}{\omega_1^*}, \quad q_7^* = \frac{\beta_3 \tau_q^1}{\omega_1^*}.$$

Now we study the propagation of plane waves in different principle plane as follows:

Case1. Let us consider plane harmonic waves propagating in a principal plane perpendicular to the principal direction (0, 1, 0) i.e. wave normal $n = (\sin \theta, 0, \cos \theta)$ inclined at angel θ to x_3-axis. The characteristic equation (26) reduces to

$$\xi^2(\delta_3 \sin^2 \theta + \delta_4 \cos^2 \theta) - \omega^2 = 0, \tag{27}$$

$$E_1 \xi^6 + E_2 \xi^4 + E_3 \xi^2 + E_4 = 0 \tag{28}$$

where

$E_1 = r_8(r_1 r_4 - r_2 r_8),$

$E_2 = r_1 \omega^2 (r_4 q_5^* - r_8) - \omega^2 (r_4 r_8 + q_5^* r_2^2)$
$\quad + r_5(r_2 r_6 - r_1 r_7) + r_3(r_2 r_7 - r_4 r_6),$

$E_3 = \omega^4 [r_8 - q_5^* (r_1 + r_4)] + \omega^2 (r_7 r_5 + r_3 r_6), \quad E_4 = \omega^6 q_5^*.$

$r_1 = \sin^2 \theta + \delta_4 \cos^2 \theta, \ r_2 = \delta_2 \sin \theta \cos \theta, \ r_3 = i\gamma_1 \sin \theta,$

$r_4 = \delta_4 \sin^2 \theta + \delta_5 \cos^2 \theta, \ r_5 = i\gamma_3 \cos \theta, r_6 = i\omega^2 q_6^* \sin \theta,$

$r_7 = i\omega^2 q_7^* \cos \theta, \ r_8 = i\omega(q_1^* \sin^2 \theta + q_2^* \cos^2 \theta) - q_3^* \sin^2 \theta - q_4^* \cos^2 \theta,$

$h_1 = r_1 r_4 - r_2 r_2, \ h_2 = r_2 r_3 - r_1 r_5, \ h_3 = r_2 r_5 - r_3 r_4, \ h_4 = r_3 r_6 - r_1 q_5^* \omega^2,$

$G_1 = \Gamma_2, \quad G_2 = (q_5 - \Gamma_2)\omega^2 - i\gamma_1 \Gamma_1, \quad G_3 = -q_5,$

$\Gamma_1 = i\omega^2 q_6^*, \quad \Gamma_2 = i\omega q_1^* - q_3^*.$

Equation (27) corresponds to purely transverse wave (SH) wave, which is not affected by thermal variations.

Case 11. For $\theta = 90^0$, i.e. when the wave normal $n = (1, 0, 0)$ is perpendicular to the x_3- axis, the equation (26)reduces to

$$\delta_3 \xi^2 - \omega^2 = 0, \delta_4 \xi^2 - \omega^2 = 0, \tag{29}$$

$$G_1 \xi^4 + G_2 \xi^2 + G_3 = 0, \tag{30}$$

where

$$G_1 = iq_1^* \omega - q_3^*, \ G_2 = \omega^2 (q_5^* - i\omega q_1^* + q_3^* + \gamma_1 q_6^*), \ G_3 = -q_5^* \omega^4.$$

Equation (29) corresponds to purely transverse waves, which are not affected by thermal variations.

6 Particular cases

1. Taking

$$\bar{c}_{11} = \bar{c}_{22} = \bar{c}_{33}, \quad \bar{c}_{12} = \bar{c}_{13}, \quad \bar{c}_{44} = \bar{c}_{66}, \quad \beta_1 = \beta_2 = \beta_3, \tag{31}$$

$$K_1 = K_3 = K, \quad K_1^* = K_3^* = K^*, \tag{32}$$

yields the corresponding results for cubic crystal materials.

2. The corresponding results for isotropic thermoviscoelastic are obtained by taking

$$\bar{c}_{11} = \bar{c}_{33} = \bar{\lambda} + 2\bar{\mu}, \quad \bar{c}_{12} = \bar{c}_{13} = \bar{\lambda}, \quad c_{44} = \bar{\mu}, \quad \beta_1 = \beta_3, \quad (33)$$

$$K_1 = K_3 = K, \quad K_1^* = K_3^* = K^*. \quad (34)$$

7   Numerical results and discussion

For numerical computations, we take the following values of the relevant parameters given as follows

$$c_{11} = 18.78 \times 10^{10} Kg.m^{-1}s^{-2}, \quad c_{12} = 6.76 \times 10^{10} Kg.m^{-1}s^{-2},$$

$$c_{13} = 8.0 \times 10^{10} Kg.m^{-1}s^{-2}, \quad c_{33} = 10.2 \times 10^{10} Kg.m^{-1}s^{-2},$$

$$c_{44} = 10.06 \times 10^{10} Kg.m^{-1}s^{-2}, \quad T_0 = 0.293 \times 10^3 K,$$

$$\alpha_1 = 1.96 \times 10^{-5} K^{-1}, \quad \alpha_3 = 1.4 \times 10^{-5} K^{-1},$$

$$K_1 = 0.12 \times 10^3 Wm^{-1}K^{-1}, \quad K_3 = 0.33 \times 10^3 Wm^{-1}K^{-1},$$

$$C^* = 0.6331 \times 10^3 JKg^{-1}K^{-1}, \quad \rho = 8.954 \times 10^3 Kg.m^3,$$

$$K_1^* = \bar{c}_{11}C^*/4, \quad K_3^* = \bar{c}_{33}C^*/4.$$

We can solve equation (28) with the help of the software Matlab 7.0.4 and after solving the equation (30) and using the formulas given in **Appendix A [A.1-A.4]**, we can commute the values of phase velocity(V_1,V_2, V_3), attenuation coefficient(Q_1,Q_2, Q_3), specific loss (W_1,W_2, W_3) and penetration depth(P_1,P_2, P_3) for intermediate values of frequency (ω) in theories of two phase and three phase lag model. The solid line corresponds to two-phase-lag model (11 phase lag), doted lines correspond to three-phase-lag model (111 phase lag) and center symbols on these lines corresponding to two-phase-lag model (11 phase lag (Vis)),(111 phase lag (Vis)) with viscous respectively.

8   Phase velocity

Figs.1,2 and 3 depict the variation of phase velocity V_1, V_2 and V_3 of waves with frequency ω. It is evident from fig.1 that the values of V_1 increase for initial values ω whereas for higher values of ω, the values of V_1 slightly
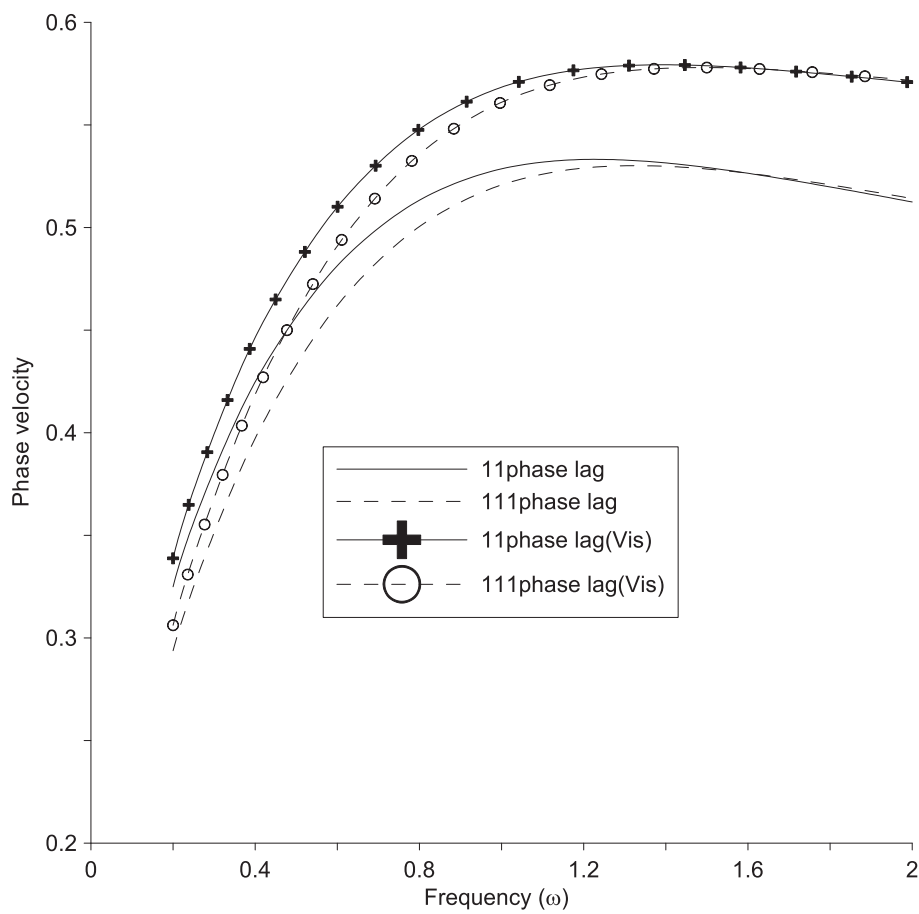
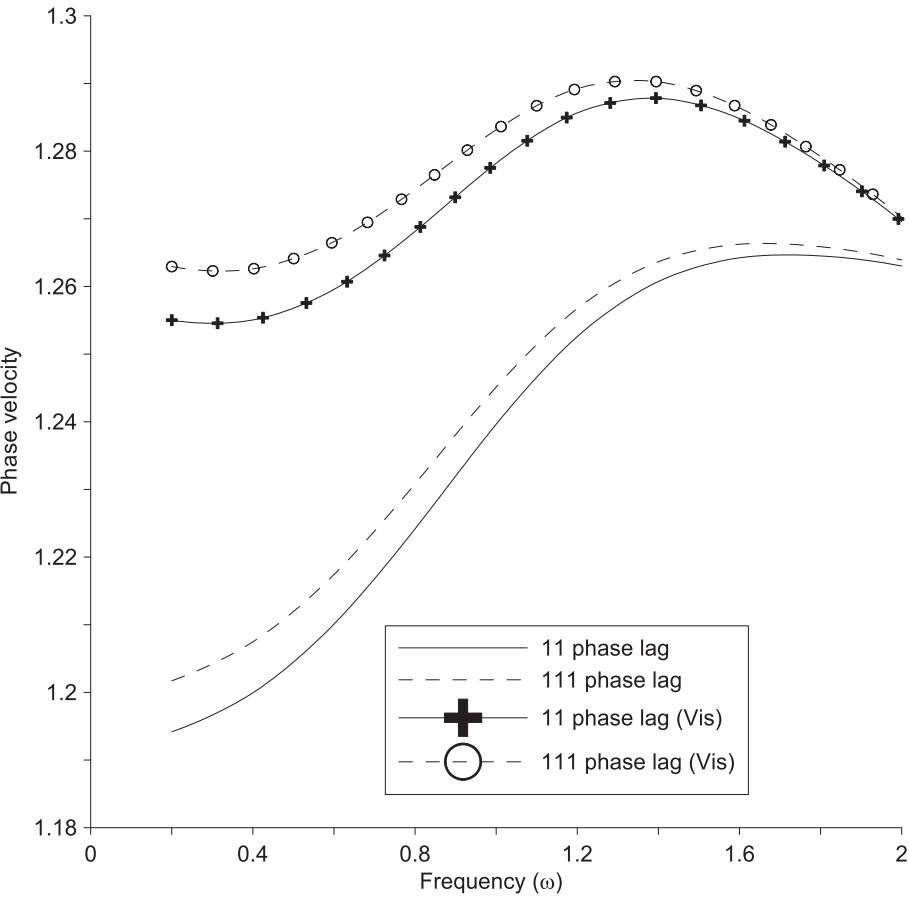Figure 1: Variation of phase velocity (V_1) with frequency

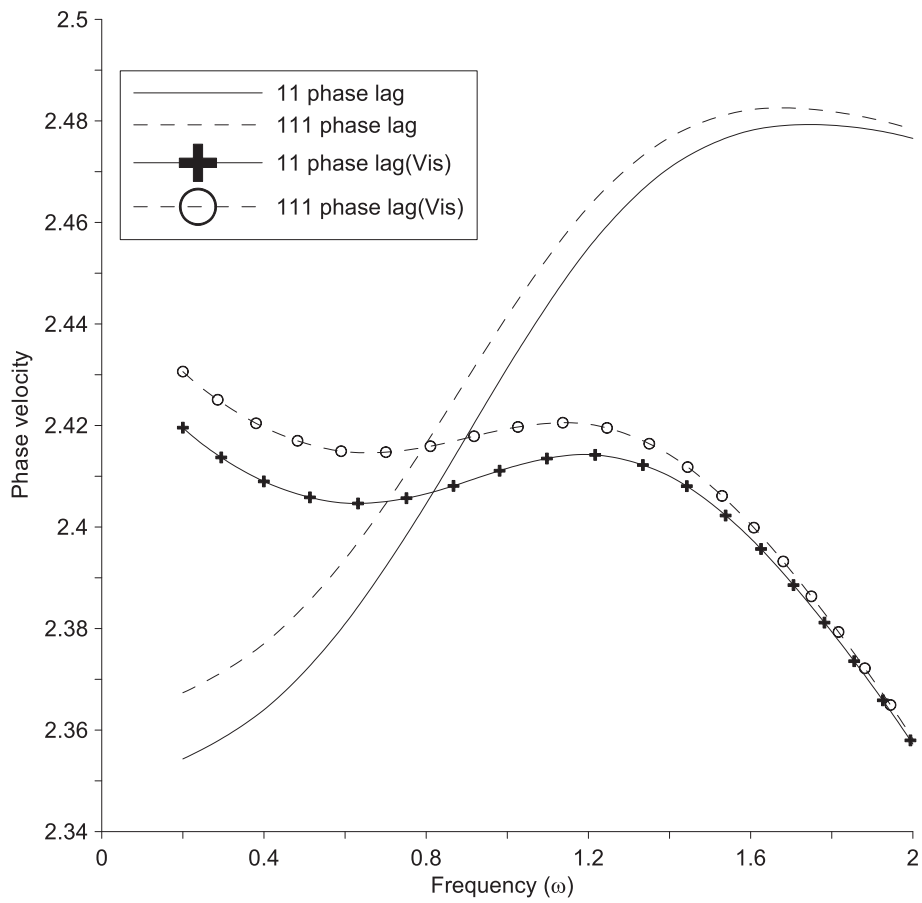Figure 2: Variation of phase velocity (V_2) with frequency

Figure 3: Variation of phase velocity (V_3) with frequency

decreases. It is noticed that due to viscosity effect, the values of V_1 remain more. Fig.2 represents that the values of V_2 increases for smaller values of ω, although for higher values of ω, the values of V_2 slightly decreases. It is evident that that the values of V_2 in case of with viscous effect remain more (In comparison with without viscous effect). Fig.3 shows that the values of V_3 in case of without viscous effect increases for smaller values ω, but for higher values of ω, the values of V_3 slightly decreases and due to viscosity effect, the values of V_3 decrease for higher values of ω. It is evident that the values of V_3 due to viscosity effect become smaller in comparison to without viscous effect for higher values of ω.

9   Attenuation coefficient and specific loss

Figs.4,5 and 6 depict the variation of attenuation coefficient (Q_1,Q_2, Q_3) of waves with frequency ω. Fig.4 shows that the values of Q_1 slightly increase for smaller values of ω, whereas for higher values of ω, the values of Q_1 decrease. It is evident that the values of Q_1 in case of without viscous effect remain more(In comparison with viscous effect). Fig.5 indicates that the values of Q_2 increase for initial values of ω although for higher values of ω, the values of Q_2. It is evident that due to viscosity effect the values of Q_2 remain more. Fig.6 exhibits the variation of Q_3 with ω and it indicates that the behavior and variation of Q_3 is same as Q_2, although the magnitude values of Q_3 are different.

Fig. 7 depicts the variation of specific loss (W_1) of waves with frequency ω. It is evident that the values of W_1 decrease for higher values of ω. If we compare the results we find that the values of W_1 in case of with viscous effect remain more (In comparison with without viscous effect).Fig. 8 shows that the values of specific loss (W_2) increases for smaller values of ω, but for higher values of ω reverse behavior occurs. It is noticed that the values of W_2 due to viscosity effect remain more. Fig.9 shows the variation of W_3 with ω and it indicates that the behavior and variation of W_3 is same as W_2, whereas the magnitude values of W_3 are different.

Penetration depth

Figs.10, 11 and 12 represent the variation of penetration depth (P_1, P_2, and $P_{3)}$ of wave with frequency ω. It is evident from fig.10 that the values P_1 increase for higher values of ω. Fig.11 shows that the values P_1 decrease for initial values of ω, but for higher value of ω the values of P_1 increase. Fig.12

Figure 4: Variation of attenuation coefficient (Q_1) with frequency

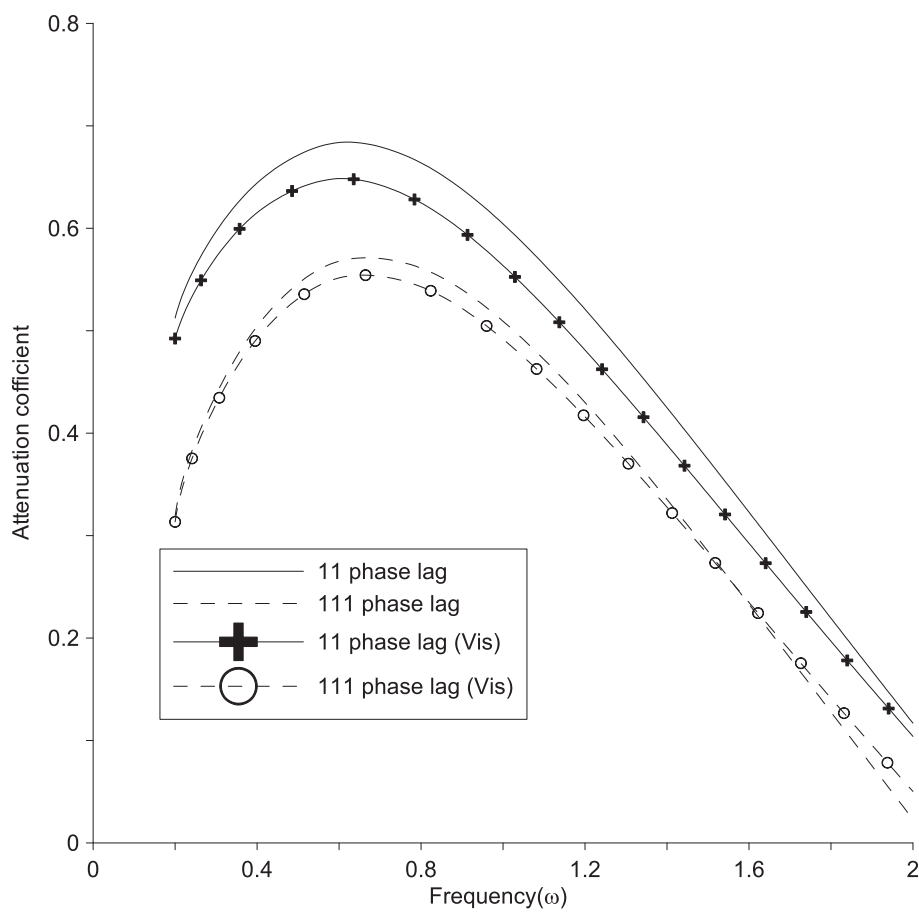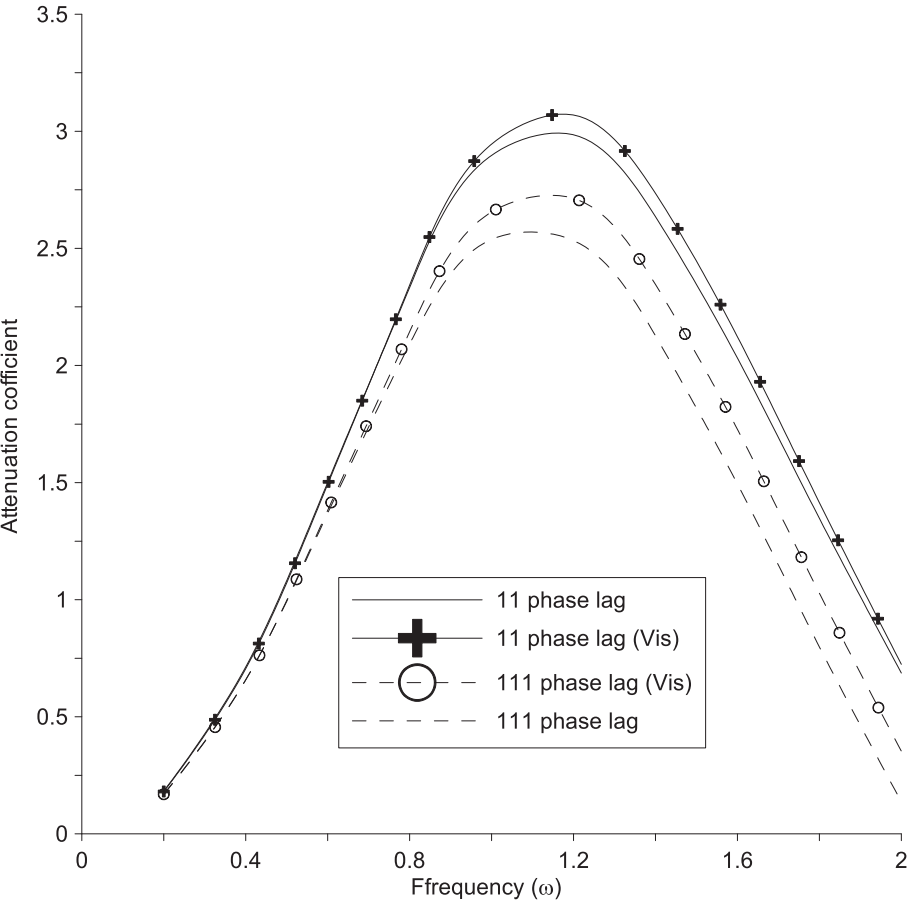*Rajneesh Kumar, Vijay Chawla, Ibrahim A.Abbas*

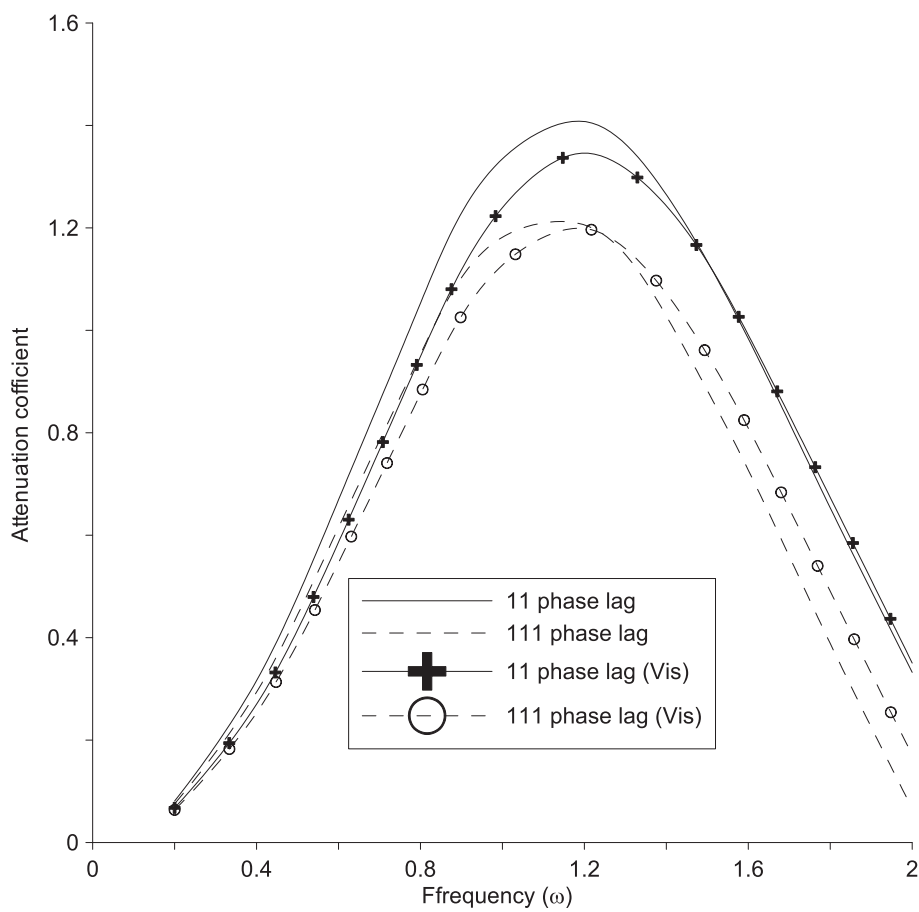Figure 5: Variation of attenuation coefficient (Q_2) with frequency

Figure 6: Variation of attenuation coefficient (Q_3) with frequency
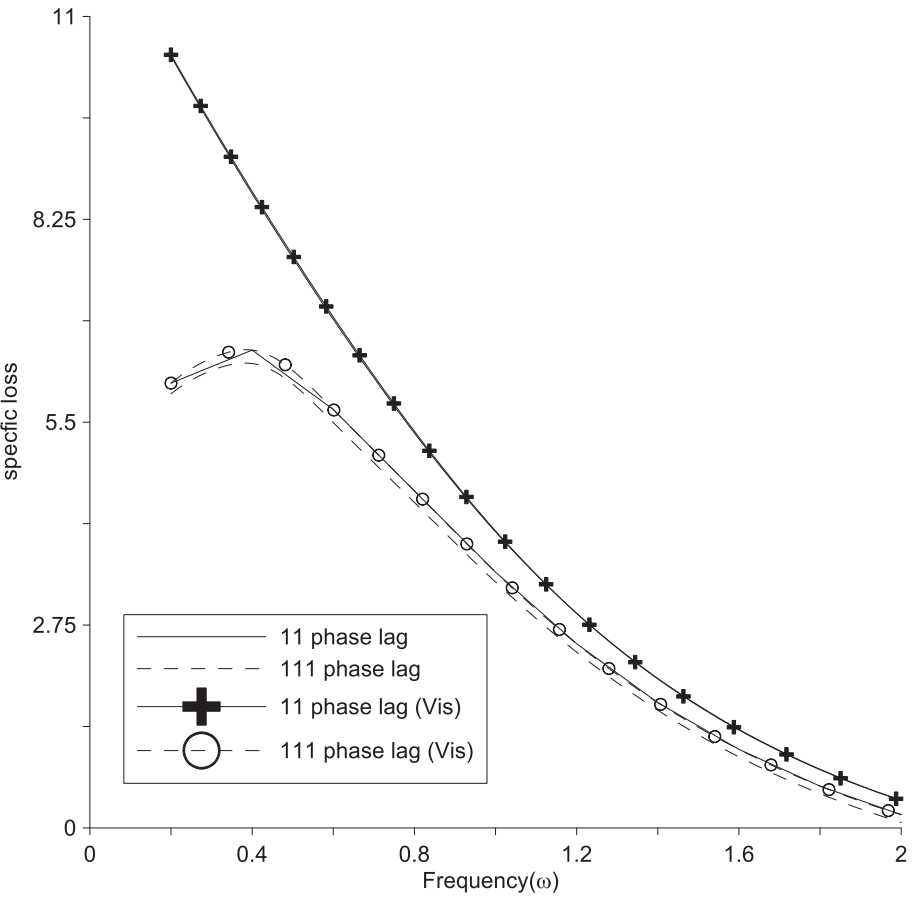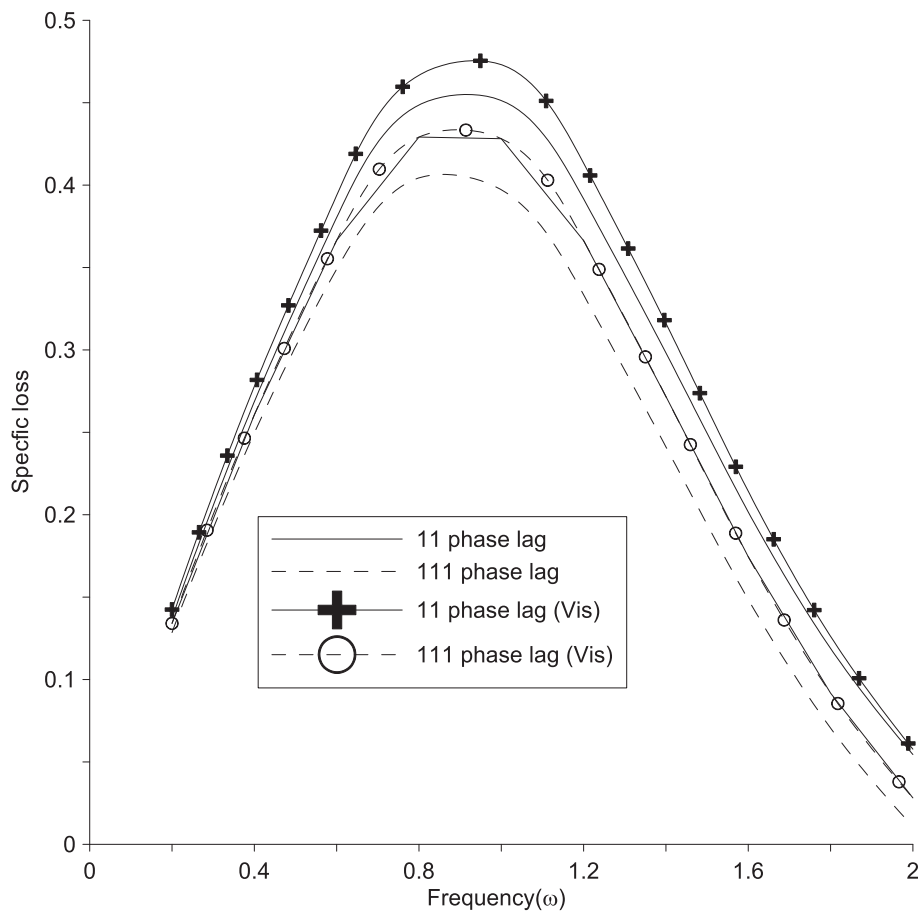
Figure 7: Variation of specific loss (W_1) with frequency

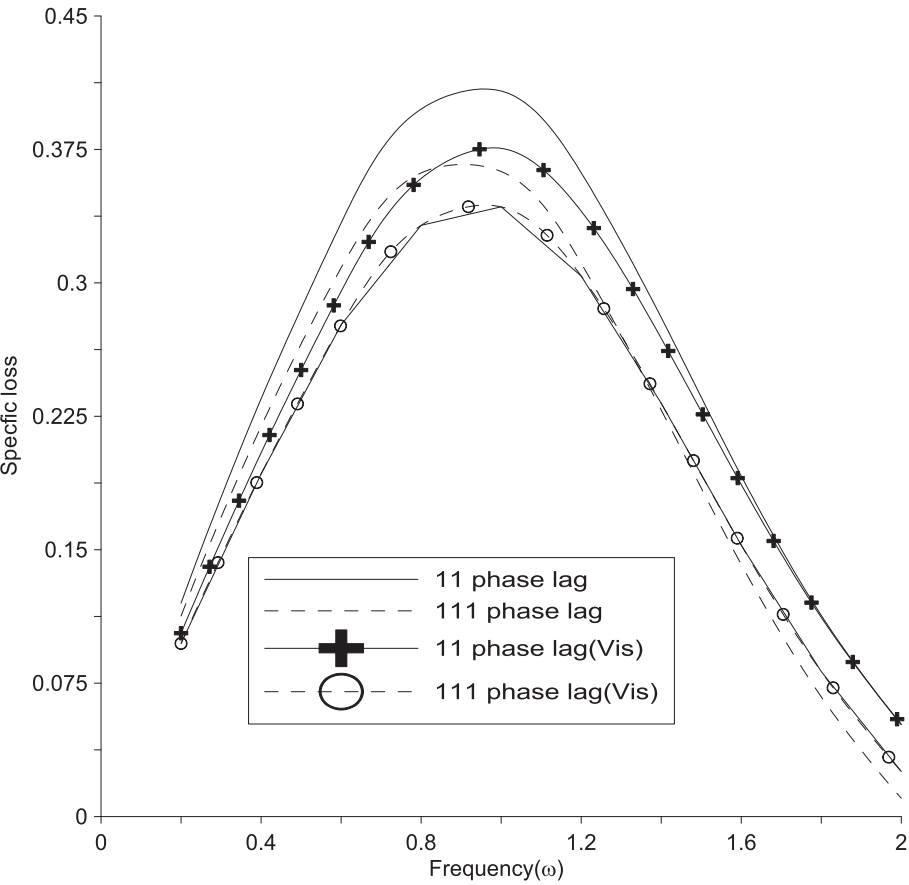Figure 8: Variation of specific loss (W_2) with frequency

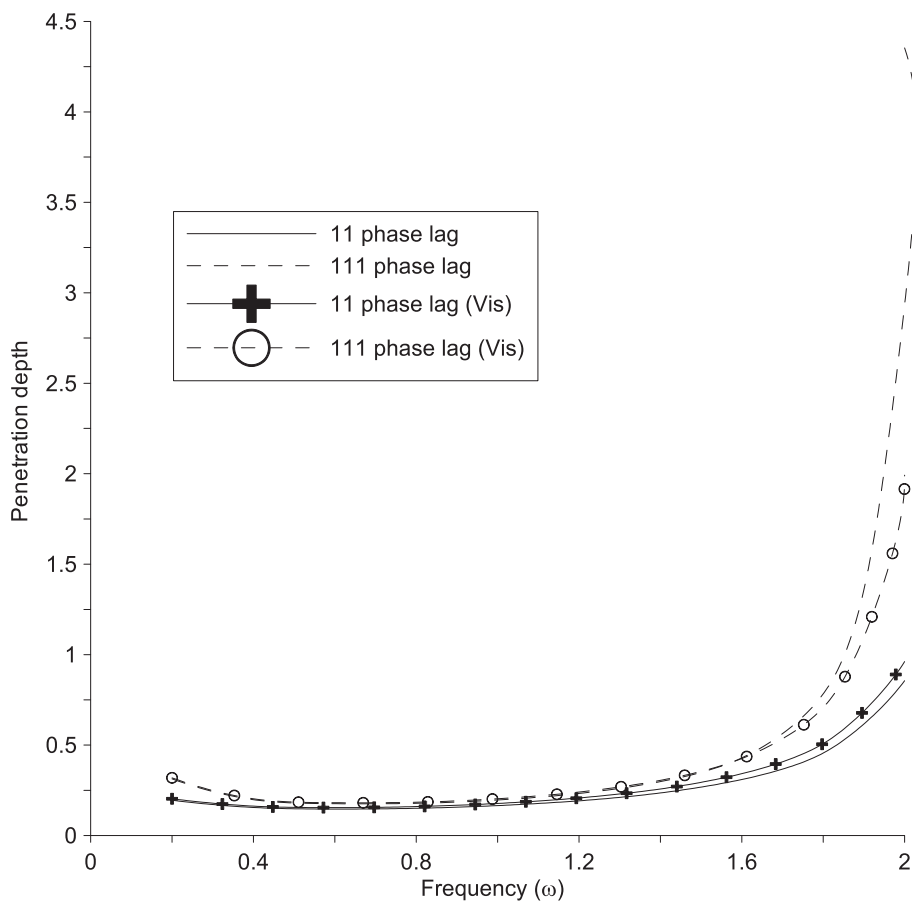Figure 9: Variation of specific loss (W_3) with frequency

Figure 10: Variation of penetration depth (P_1) with frequency

depicts the variation of penetration depth (P_3) with ω. It is noticed that due to viscosity effect, the values of P_1, P_2, P_3 are smaller (in comparison to without viscous effect).

Concluding remarks

The propagation of waves in anisotropic thermoviscoelastic medium in the context of the theory of three-phase-lag model has been studied. The governing equations for homogeneous transversely isotropic thermoviscoelastic three-phase-lag are reduced as a special case. When plane waves propragate in a principle plane of transversely thermoviscoelastic three-phase-lag model,
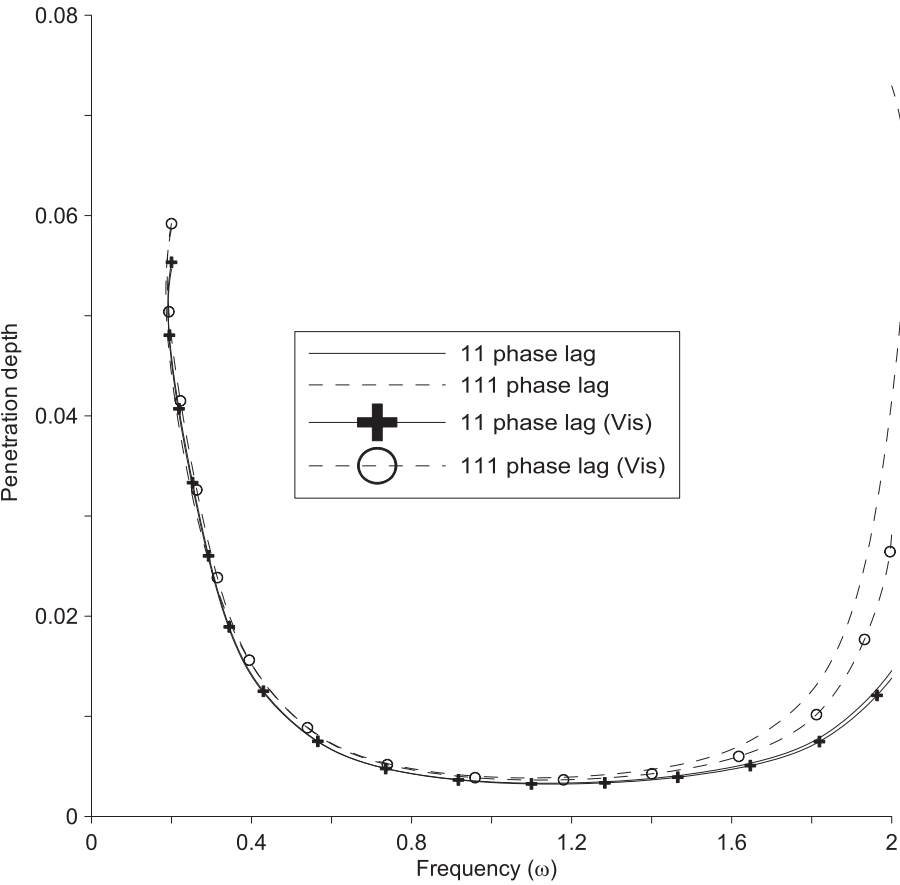
Figure 11: Variation of penetration depth (P_2) with frequency
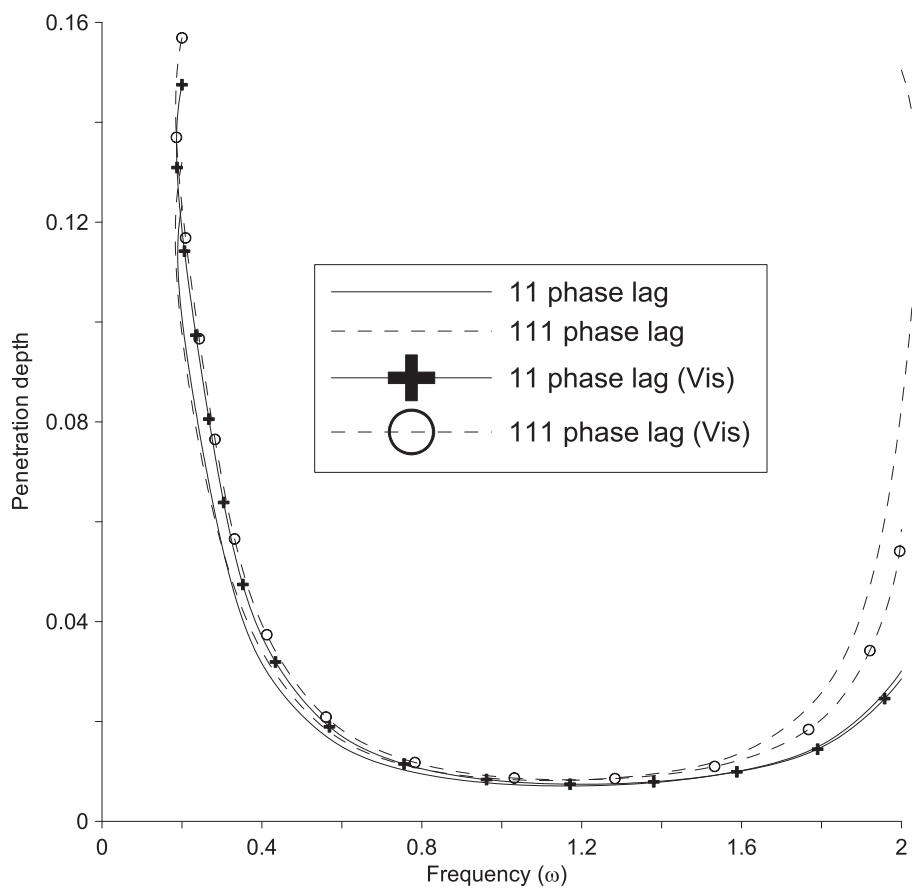
Figure 12: Variation of penetration depth (P_3) with frequency

purely transverse wave mode decouple from the rest of the motion and is not affected by the thermal variation. In case of plane wave propagation along the axis of solid, two purely transverse wave modes decouples from the rest of motion and not affected by the thermal vibration. Different characteristics of waves like phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and presented graphically.

From numerical and graphical results, it is clear that due to viscosity effect, the values of phase velocity V_1, V_2, attenuation coefficient Q_2, specific loss (W_1, W_2) remain more, whereas for the case of without viscous effect the values of phase velocity V_3, attenuation coefficient (Q_1, Q_3), specific loss (W_3) and penetration depth (P_1, P_2, P_3) remain more.

Appendix A

(i) Phase velocity

The phase velocities are given by

$$V_i = \frac{\omega}{Re(\xi_i)}, \quad i = 1, 2, 3, 4, \tag{A.1}$$

where V_1, V_2, V_3, V_4 are the velocities of $qP1, qS1, qS2$ and $qP2$ waves respectively.

(ii) Attenuation Coefficient

The attenuation coefficient is defined as

$$Q_i = Img(\xi_i), \quad i = 1, 2, 3, 4, \tag{A.2}$$

where $Q_i, \ i = 1, 2, 3, 4$ are the attenuation coefficients of $qP1, qS1, qS2 and qP2$ waves respectively.

(iii) Specific Loss

The specific loss is the ratio of energy (ΔW) dissipated in taking a specimen through cycle, to elastic energy (W) stored in a specimen when the strain is maximum. The specific loss is the most direct method of defining internal friction for a material. For a sinusoidal plane wave of small amplitude it was shown by Kolsky [25] that specific loss $\Delta W/W$ equals 4π times the absolute value of the imaginary part of ξ to the real part of ξ i.e.

$$W_i = \left(\frac{\Delta W}{W}\right) i = 4\pi \left|\frac{Img(\xi_i)}{Re(\xi_i)}\right|, \quad i = 1, 2, 3, 4. \tag{A.3}$$

(iv) Penetration depth
The penetration depth is defined by

$$B_i = \frac{1}{Img(\xi_i)}, \quad i = 1, 2, 3, 4. \tag{A.4}$$

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Uticaj viskoznosti na prostiranje talasa u anizotropnoj termoelastičnoj sredini sa modelom trofaznog zaostajanja

Proučava se prostiranje talasa u anizotropnoj viskoelastičnoj sredini u kontekstu teorije trofaznog zaostajanja termoelastičnih matrerijala. Nadjena su dva kvaziuzdužna talasa $(qP1, qP2)$ i dva poprečna talasa $(qS1, qS2)$. Vodeće jednačine za homogenu poprečno izotropnu termoviskoelastičnu sredinu su redukovane kao poseban slučaj posmatranog modela. Razne karakteristike talasa kao: fazna brzina, koeficijent slabljenja, specifični gubitak i dubina prodiranja su izračunate iz dobijenih rezultata. Viskozni efekt je pokazan grafički za različite rezultujuće veličine za modele dvofaznog zaostajanja i trofaznog zaostajanja termoelastičnosti. Neki posebni značajni slučajevi su takodje izvedeni iz datog istraživanja.